Density of rational points on Enriques surfaces

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1 Introduction

Let X be an irreducible algebraic variety defined over a number field K. We will say that rational points on X are potentially dense if there exists a finite extension K' of K such that the set of K'-rational points is Zariski–dense in X. It seems reasonable to ask whether or not rational points are potentially dense if neither X nor its unramified coverings admit a morphism onto a variety of general type. This question has an easy answer if the dimension of X equals 1. Very little is known in higher dimensions. Clearly, potential density holds for unirational varieties and for abelian varieties. In [4] and [3] it is proved that rational points are potentially dense on all Fano threefolds with the possible exception of double covers of $\mathbb{P}^3$ ramified in smooth surfaces of degree 6.

In this paper we will study the question of density under the additional assumption that X admits the structure of an elliptic fibration $\varphi : X \to B$ over some irreducible normal base B of dimension ≥ 1. As one of the applications we prove that rational points on Enriques surfaces are potentially dense. An alternative idea to prove density would have been to use the group of automorphisms $\mathrm{Aut}(X)$. Notice, however, that there exist Enriques surfaces with a finite group $\mathrm{Aut}(X)$ (cf. [1]).

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2 Elliptic fibrations - Generalities

Let K be a number field and X a smooth projective algebraic surface defined over K. We say that X admits a structure of an elliptic fibration if there exists a regular map $\varphi : X \to B$ onto a smooth (irreducible) curve B whose fibers are connected curves such that the generic fiber is a smooth curve of genus 1. We denote by X_b the fiber over $b \in B$. We will say that X admits a structure of a Jacobian elliptic fibration if there exists a zero section $e : B \to$

X. It is well known (cf. [2] or [5]) that to every elliptic fibration $\varphi : \mathcal{E} \to B$ one can associate a Jacobian elliptic fibration $\varphi_{\mathcal{J}} : \mathcal{J} = \mathcal{J}(\mathcal{E}) \to B$ (over the same groundfield), which over the generic point of the base B is given by classes of divisors of degree zero in the fiber. The zero section corresponds to the trivial class. There is a fiberwise action of $\mathcal{J}$ on $\mathcal{E}$, more precisely, a rational map

$$\psi : \mathcal{J} \times_B \mathcal{E} \to \mathcal{E}$$

which is regular in non-singular points of the fibers of $\mathcal{J}$ and $\mathcal{E}$ and which induces a transitive action of $\mathcal{J}_b$ on $\mathcal{E}_b$ (for smooth fibers).

2.1 Multisections

Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration. A multisection $i : \mathcal{M} \hookrightarrow \mathcal{E}$ is an irreducible subvariety of $\mathcal{E}$ such that the map $\varphi \circ i : \mathcal{M} \to B$ is finite and surjective. We will denote by $d = d(\mathcal{M})$ the degree of this projection.

Definition 2.1 Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration. A multisection $\mathcal{M}$ is said to be of *order* m if m is the smallest positive integer such that for any $b \in B$ and any pair of points $p_b, p'_b \in \mathcal{M} \cap \mathcal{E}_b$ the image of the zero-cycle $p_b - p'_b$ in $\mathcal{J}_b$ is torsion of order m.

Note that *any* section of an elliptic fibration is a multisection of order one.

Let $\Phi_m \subset \mathcal{J}$ be the subvariety of of m-torsion points of $\mathcal{J}$ (all points p_b such that $m \cdot p_b = 0$ in the group of rational points of the corresponding fiber $\mathcal{J}_b$) which are not contained in the zero section $e(B)$.

Lemma 2.2 *If $\mathcal{M} \subset \psi(\Phi_m \times_B \mathcal{M})$ then $\mathcal{M}$ is a multisection of order m.*

Proof. Tautology.

Definition 2.3 Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration. A *saliently ramified multisection* of $\mathcal{E}$ is a multisection $\mathcal{M}$ which is ramified in a point p_b which lies in a smooth (elliptic curve) fiber $\mathcal{E}_b$ of $\mathcal{E}$.

Proposition 2.4 *Let $\mathcal{M}$ be a saliently ramified multisection of $\mathcal{E}$. Then $\mathcal{M}$ is not of order m for any $m > 0$.*

Proof. By assumption, nearby fibers of $\mathcal{E}_b$ are smooth. Consider an embedding $K \hookrightarrow \mathbb{C}$. We can find a sequence $(b_i) \in B(\mathbb{C})$ of points in the base converging to b (in complex topology) and pairs of distinct points p_i, p_i' in the fibers $\mathcal{E}_{b_i}(\mathbb{C})$ which converge to p. If we assume that the cycle $p_i - p_i'$ is torsion of order m (for some $m \geq 2$) in the Jacobian we obtain a contradiction since it converges to $0 = e(b)$.

Definition 2.5 Let $\mathcal{M}$ be a multisection of degree d of the elliptic fibration $\varphi : \mathcal{E} \to B$ and $b \in B$. We denote by $\mathrm{Tr}_{\mathcal{M}}(b)$ the zero-cycle $\mathcal{M} \cap \mathcal{E}_b$. Define the class map

$$\tau_{\mathcal{M}} \, : \, \mathcal{E}_B \to \mathcal{J}_B$$

by the following rule:

$$\tau_{\mathcal{M}}(p) = [d \cdot p - \mathrm{Tr}_{\mathcal{M}}(\varphi(p))]$$

for $p \in \mathcal{E}$.

Lemma 2.6 *Suppose that the multisection $\mathcal{M}$ is of degree d and not of order d' with $d'|d$. Then the map*

$$\tau_{\mathcal{M}} \, : \, \mathcal{M} \to \tau_{\mathcal{M}}(\mathcal{M})$$

is a birational isomorphism.

Proof. Consider all points $p, p' \in \mathcal{E}$ such that $\tau_{\mathcal{M}}(p) = \tau_{\mathcal{M}}(p')$. Then the cycle $p - p'$ is of order $d'|d$ in $\mathcal{J}_b$. On the other hand, for a given d' pairs of distinct points differing by torsion of order d' (more precisely, by a translation by $\Phi_{d'}$) constitute a Weil divisor in $\mathcal{M}$. Therefore, the map $\tau_{\mathcal{M}}$ is a birational isomorphism.

Proposition 2.7 *The map $\tau_{\mathcal{M}}$ is regular outside of singular fibers of $\mathcal{E} \to B$.*

Proof. Evident.

Corollary 2.8 *Let $\mathcal{M}$ be a saliently ramified multisection of $\mathcal{E}$. Then $\tau_{\mathcal{M}}(\mathcal{M})$ is a saliently ramified multisection of $\mathcal{J}$.*

Proof. The map $\tau_{\mathcal{M}}$ is unramified on the set of non-singular fibers. By lemma 2.6, it is a birational isomorphism.

Remark 2.9 Let $\mathcal{M} \subset \mathcal{E}$ be a multisection which is not of order m for any m. Then $\tau_{\mathcal{M}}(\mathcal{M})$ is not of order m for any m.

From now on we will restict to the case of the base $B = \mathbb{P}^1$. Merel's theorem implies:

Proposition 2.10 *[3] Let $\varphi_{\mathcal{J}} : \mathcal{J} \to \mathbb{P}^1$ be a Jacobian elliptic fibration defined over K with a saliently ramified multisection $\mathcal{M}$. Then for all but finitely many $b \in \varphi_{\mathcal{J}}(\mathcal{M}(K)) \subset \mathbb{P}^1(K)$ the fibers J_b have infinitely many rational points.*

Theorem 2.11 *Let $\varphi : \mathcal{E} \to \mathbb{P}^1$ be an elliptic fibration defined over K. Assume that there exists a rational or elliptic saliently ramified multisection $\mathcal{M}$ defined over K. Then rational points on $\mathcal{E}$ are potentially dense.*

Proof. Extending the groundfield, we can assume that $\mathcal{M}$ is defined over K, that K-rational points on $\mathcal{M}$ are Zariski-dense and that it intersects a smooth fiber $\mathcal{E}_{\varphi(p)}$ with local intersection multiplicity ≥ 2 at a K-rational point p. The image $\varphi(\mathcal{M}(K))$ is Zariski-dense in $\mathbb{P}^1$. Every fiber $\mathcal{E}_b$ for $b \in \varphi(\mathcal{M}(K))$ contains at least one point $p_b \in \mathcal{M}(K)$. Therefore, it suffices to show that for almost all $b \in \varphi(\mathcal{M}(K))$ the fiber $\mathcal{J}_b$ has a K-rational point of infinite order. Then we use the action of the $\mathcal{J}$ on $\mathcal{E}$ to translate p_b.

By assumption and 2.8 the curve $\tau_{\mathcal{M}}(\mathcal{M})$ is a saliently ramified multisection of $\mathcal{J} \to \mathbb{P}^1$. The point $\tau_{\mathcal{M}}(p_b) \in \tau_{\mathcal{M}}(\mathcal{M})$ is a K-rational point of the fiber $\mathcal{J}_{\varphi(p_b)}$. Moreover, it is contained in the saliently ramified multisection $\tau_{\mathcal{M}}(\mathcal{M})$. By theorem 2.10, for all but finitely many fibers $\mathcal{J}_{\varphi(p_b)}$ the point $\tau_{\mathcal{M}}(p_b)$ is a non-torsion point in the group $\mathcal{J}_{\varphi(p_b)}(K)$. This concludes the proof.

Remark 2.12 An alternative argument avoiding Merel's theorem goes as follows: We can find a base change $\beta : B' \to B$ with the following properties: β is étale at b (corresponding to the smooth fiber of $\mathcal{E} \to B$ where $\mathcal{M}$ is ramified), $\mathcal{M}$ pulls back to a section $\mathcal{M}'$ of $\mathcal{E}' \to B'$ and $\mathcal{E}'$ acquires a zero section e' (which is different from $\mathcal{M}'$). Then $\mathcal{M}'$ must be of infinite order in the Mordell–Weil group of $\mathcal{E}'$. A specialization argument (cf. [7]) implies that rational points are (potentially) dense on $\mathcal{E}'$.

Proposition 2.13 *Let X be a smooth algebraic surface defined over a number field K and admitting two different elliptic fibrations $\varphi_1 : \mathcal{E}_1 \to B_1$ and*

$\varphi_2 : \mathcal{E}_2 \to B_2$ *with* $B_1 \simeq B_2 \simeq \mathbb{P}^1$*. Then rational points on* X *are potentially dense.*

Proof. If there is at least one $\mathcal{E}_{b_2}$ which is not of order m for any m we are done by 2.9. In view of 2.8, it suffices to consider the case when for all $b_2 \in B_2$ all multiple intersection points of the multisections $\mathcal{E}_{b_2}$ with fibers $\mathcal{E}_{b_1}$ (for $b_1 \in B_1$) are contained in the singular fibers of the fibration $\mathcal{E}_1 \to B_1$. If a generic $\mathcal{E}_{b_2}$ is of some fixed order m_0 we see that (a cover of) X is dominated (birationally) by a product of two elliptic curves.

3 Enriques surfaces

We start with a brief summary of the structure theory of Enriques surfaces (cf. [2], pp. 274-275): Either X has two distinct elliptic fibrations over $\mathbb{P}^1$ (this case is called *non-special* in [2]) or it admits an elliptic fibration with a 2-section which is a (-2)-curve. In the second case (called *special*) the associated K3 double cover $\pi : Y \to X$ admits the structure of an elliptic fibration $Y \to \mathbb{P}^1$ with *two* (non-intersecting) sections (which could differ by torsion). The surface Y is the minimal resolution of a double covering of a quadratic cone $Q \subset \mathbb{P}^3$ (given by $z_0 z_1 = z_2^2$ in standard coordinates in $\mathbb{P}^3 = (z_0, z_1, z_2, z_3)$) ramified in an intersection of this cone with a quartic hypersurface in $\mathbb{P}^3$ not passing through the vertex of this cone. The ramification curve R is reduced, of degree 8 and has at most simple singularities. The fibers of the elliptic fibration correspond to the generators of the cone Q and the two sections are mapped to the vertex of Q (cf. [2], p. 278).

Theorem 3.1 *Let* X *be an Enriques surface over* K*. Then rational points on* X *are potentially dense.*

Proof. In the non-special case we apply 2.13 and we are done.

Now let us consider the special case. Recall that the elliptic fibration $Y \to \mathbb{P}^1$ has two non-intersecting sections e_1, e_2. Their difference is torsion in the Picard group of the generic point of $Y \to \mathbb{P}^1$ if there exists an integer $m > 0$ such that $m(e_1 - e_2)$ can be represented as a sum, with integer coefficients, of components of singular fibers of $Y \to \mathbb{P}^1$. If all fibers of $Y \to \mathbb{P}^1$ are irreducible then the difference $e_1 - e_2$ is not torsion and rational points are potentially dense. Otherwise, we have to consider subcases of the special case.

After blowing up the vertex of the cone we can realize Q birationally as $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2) \oplus \mathcal{O})$ (see [3]). Taking sections of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2))$ (a 3-dimensional linear space) we obtain a family of (conics) $\mathbb{P}^1_s \simeq \mathbb{P}^1 \subset Q$ where each $\mathbb{P}^1_s$ intersects the ramification curve R in 8 points (counted with multiplicities). The double cover D_s of $\mathbb{P}^1_s$ is a multisection of the elliptic fibration $Y \to \mathbb{P}^1$. We want to find an elliptic saliently ramified multisection among the D_s. Then we apply 2.11.

Denote by $R^0 \subset R$ the Zariski open subset of points where the curve R is smooth and where it is not tangent to the generators of the cone Q. For every point $P \in R^0$ we consider the affine line L_P of sections $\mathbb{P}^1_s$ which are tangent to R at P. These sections cover the whole cone Q, except the line joining P and the vertex of Q. Consider the subset of sections of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2))$ which have at least two distinct points of local intersection multiplicity ≥ 2 with R. If this subset covers birationally the cone Q, then we obtain a 1-dimensional family of elliptic curves which covers Y and which is generically transversal to the elliptic fibration $Y \to \mathbb{P}^1$. Hence we can apply 2.13.

In particular, if the curve R has a singular double point $r \in R$, then for any $P \in R^0$ we can find a tangent section $\mathbb{P}^1_s$ which passes through r. The family of such sections covers Q (birationally), provided that R has a component which is not a section. Therefore, we have only to consider the case when R consists of 4 distinct components which are sections. Suppose that r lies on the intersection of two components. Then the tangents to some point of a third component passing through r cover Q birationally and the argument above applies.

The only remaining case is the case of a non-singular curve R. For any $P \in R$ there exists a section $s \in L_P$ which has local intersection multiplicity ≥ 2 with R at some other point. Indeed, we have a natural tangent correspondence $\mathrm{TC}_P \subset L_P \times R$ defined by

$$\mathrm{TC}_P = \{(s, r) \mid s \in L_P, r \in \mathbb{P}^1_s \cap R\}.$$

Since R is irreducible this correspondence is irreducible. Since R is not a rational curve this correspondence has ramification points over L_P. Ramification points correspond to sections having at least two intersection points of local multiplicity ≥ 2 with R. Hence, the family of sections with this property (as P moves over R^0) covers (birationally) the cone Q and defines a transversal elliptic fibration on $Y \to \mathbb{P}^1$.

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